

Transfinite thin plate spline interpolation

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Abstract

Duchon's method of thin plate splines defines a polyharmonic interpolant to scattered data values as the minimizer of a certain integral functional. For transfinite interpolation, i.e. interpolation of continuous data prescribed on curves or hypersurfaces, Kounchev has developed the method of polysplines, which are piecewise polyharmonic functions of fixed smoothness across the given hypersurfaces and satisfy some boundary conditions. Recently, Bejancu has introduced boundary conditions of Beppo Levi type to construct a semi-cardinal model for polyspline interpolation to data on an infinite set of parallel hyperplanes. The present paper proves that, for periodic data on a finite set of parallel hyperplanes, the polyspline interpolant satisfying Beppo Levi boundary conditions is in fact a thin plate spline, i.e. it minimizes a Duchon type functional.

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1 Introduction

Thin plate spline interpolation, one of the main algorithms for multivariable scattered data approximation, was introduced by Duchon in [5]-[6] and has since generated a vast amount of research on its theory and applications (for a comprehensive survey, see Wendland [18]). Among all functions $F : \mathbb{R}^d \rightarrow \mathbb{R}$ having all partial derivatives $\partial^\alpha F$ of total order $|\alpha| = p$ in $L^2(\mathbb{R}^d)$ (where $p > d/2$) and taking prescribed values at a finite (sufficiently large) number of fixed scattered points in \mathbb{R}^d , the *thin plate spline* (or *surface spline*) interpolant to the data is defined as the unique minimizer of the integral functional

$$\int_{\mathbb{R}^d} \sum_{|\alpha|=p} \frac{p!}{\alpha!} |\partial^\alpha F(x)|^2 dx. \quad (1)$$

Duchon's theory identifies this minimizer as a function $S \in C^{2p-d-1}(\mathbb{R}^d)$ which is polyharmonic of order p , i.e. $\Delta^p S(x) = 0$ with Δ the Laplace operator, for all $x \in \mathbb{R}^d$ except the given scattered points. The computational usefulness of S stems from its explicit radial basis representation as a finite linear combination of translates of the fundamental solution of Δ^p in \mathbb{R}^d .

A different interpolation problem that can be formulated in the multivariable case is that of constructing a surface or function that matches continuous data prescribed on some collection of curves or hypersurfaces. In computer aided design, this problem is referred to as *transfinite interpolation* (Sabin [13]). Typical examples include the reconstruction of 2D surfaces from level curves or from track data, as well as the visualization of 3D objects from scan data. A variational solution for transfinite interpolation along a set of curves in \mathbb{R}^2 is defined by Apprato and Arcangéli [1] (see also [2, Chapter X]), and its approximate computation is obtained by smoothing finite elements. On the other hand, Kounchev's polyspline method treated in the monograph [9] is based on explicit piecewise polyharmonic functions of any number of variables.

The present paper studies the polyspline method in a simplified setting, namely for periodic data given on parallel hyperplanes. Let n, p, N be positive integers with $2 \leq p \leq N+1$, and $\tau := \{t_0, \dots, t_N\}$, where $t_0 < t_1 < \dots < t_N$ are fixed real numbers. Let Ω be the closure in \mathbb{R}^{n+1} of the union of the open strips

$$\Omega_j := \{(t, y) \in \mathbb{R} \times \mathbb{R}^n : t \in (t_{j-1}, t_j)\}, \quad j \in \{1, 2, \dots, N\}. \quad (2)$$

A function $S : \Omega \rightarrow \mathbb{C}$ is called a *polyspline* of order p on strips determined by τ in Ω if $S \in C^{2p-2}(\Omega)$, S is polyharmonic of order p on each open strip Ω_j , $j \in \{1, \dots, N\}$, and S is 2π -periodic in each of its last n variables. In order to determine a unique polyspline that interpolates continuous periodic data prescribed on the hyperplanes $\{t_j\} \times \mathbb{R}^n$, $j \in \{0, 1, \dots, N\}$, additional boundary conditions are imposed on $\{t_0\} \times \mathbb{R}^n$ and $\{t_N\} \times \mathbb{R}^n$. Then, provided that p is an even integer, Kounchev's theory [9, Theorem 20.14] identifies the polyspline interpolant as minimizer of the functional

$$\int_{[t_0, t_N] \times \mathbb{T}^n} \left| \Delta^{p/2} F(t, y) \right|^2 dt dy, \quad (3)$$

subject to the boundary and interpolation conditions, where Δ is now the Laplace operator in \mathbb{R}^{n+1} , and $\mathbb{T} := [-\pi, \pi]$.

Recently, Bejancu [3] has introduced boundary conditions of Beppo Levi type in the construction of semi-cardinal polyspline interpolation to data on the infinite set of parallel hyperplanes $\{j\} \times \mathbb{R}^n$, $j \in \{0, 1, \dots\}$. To incorporate such conditions in the above setting, let $t_{-1} := -\infty$, $t_{N+1} := \infty$, and let the corresponding open strips Ω_0, Ω_{N+1} be defined as in (2).

Definition 1.1 *Let $p \geq 2$ be a fixed integer. A function $S : \mathbb{R}^{n+1} \rightarrow \mathbb{C}$ is called a Beppo Levi polyspline of order p on strips determined by τ in \mathbb{R}^{n+1} if the following conditions hold:*

- (i) $S \in C^{2p-2}(\mathbb{R}^{n+1})$;
- (ii) S is polyharmonic of order p on each open strip Ω_j , $j \in \{0, 1, \dots, N+1\}$;
- (iii) S is 2π -periodic in each of its last n variables;
- (iv) S satisfies the Beppo Levi conditions

$$\partial^\alpha S \in L^2(\mathbb{R} \times \mathbb{T}^n), \quad \forall |\alpha| = p. \quad (4)$$

The space of all Beppo Levi polysplines of order p on strips determined by τ in \mathbb{R}^{n+1} is denoted by $\mathcal{S}_p(\tau, n)$.

The main results of the present paper (obtained in section 3) prove that, for periodic data functions prescribed on $\{t_j\} \times \mathbb{R}^n$, $j \in \{0, 1, \dots, N\}$, there exists a unique interpolant in $\mathcal{S}_p(\tau, n)$, which minimizes the Duchon type functional

$$\int_{\mathbb{R} \times \mathbb{T}^n} \sum_{|\alpha|=p} \frac{p!}{\alpha!} |\partial^\alpha F(t, y)|^2 dt dy. \quad (5)$$

This shows that the Beppo Levi polyspline interpolant is a genuine thin plate spline analog for transfinite interpolation. Moreover, our variational characterization is valid for any order $p \geq 2$, without the restriction that $p > d/2$ as in (1) or that p is even as in (3). In the excluded case $p = 1$, the results degenerate to the classical existence, uniqueness, and variational characterization of harmonic solutions to the Dirichlet problem in each strip separately.

Note that the periodicity assumption is of importance for both theoretical and practical purposes. In our proofs, it will serve to simplify certain technical arguments that guarantee the uniqueness of the polyspline constructions. On the other hand, periodic polysplines are directly applicable to the problem of transfinite interpolation on a cylinder domain, as demonstrated, for example, in the case of visualization of the heart surface in medical imaging [10].

The construction of periodic polysplines on strips is reduced to that of a family of exponential \mathcal{L} -splines by separation of variables. To describe this procedure in our setting, let $S \in \mathcal{S}_p(\tau, n)$ and, for each $t \in \mathbb{R}$, define the sequence of Fourier coefficients of S with respect to its last n variables by

$$\widehat{S}_\xi(t) := \int_{\mathbb{T}^n} e^{-i\langle \xi, y \rangle} S(t, y) dy, \quad \xi \in \mathbb{Z}^n,$$

where $\langle \xi, y \rangle$ is the dot product in \mathbb{R}^n . For a fixed $\xi \in \mathbb{Z}^n$, note that condition (i) above implies $\widehat{S}_\xi \in C^{2p-2}(\mathbb{R})$. Further, let $|\xi|$ the Euclidean norm of ξ in \mathbb{R}^n and consider the ordinary differential operator

$$\mathcal{L}_\xi := \left(\frac{d^2}{dt^2} - |\xi|^2 \right)^p, \quad (6)$$

with null-space

$$\text{Ker} \mathcal{L}_\xi = \begin{cases} \text{span} \{ e^{\pm|\xi|t}, t e^{\pm|\xi|t}, \dots, t^{p-1} e^{\pm|\xi|t} \}, & \text{if } \xi \neq 0, \\ \text{span} \{ 1, t, \dots, t^{2p-1} \}, & \text{if } \xi = 0. \end{cases}$$

From the polyharmonic condition (ii), we deduce (as in [4, Lemma 2.1])

$$\mathcal{L}_\xi \widehat{S}_\xi(t) = 0, \quad \forall t \in (t_{j-1}, t_j), \quad \forall j \in \{0, 1, \dots, N+1\}.$$

Also, the Beppo Levi conditions (4) imply (as in [3, Eq. (4)])

$$\widehat{S}_\xi \in \begin{cases} \text{Ker} \left(\frac{d}{dt} - |\xi| \right)^p & \text{on the interval } (-\infty, t_0), \\ \text{Ker} \left(\frac{d}{dt} + |\xi| \right)^p & \text{on the interval } (t_N, \infty). \end{cases}$$

These observations motivate the following definition.

Definition 1.2 For a fixed $\xi \in \mathbb{Z}^n$, the function $s : \mathbb{R} \rightarrow \mathbb{C}$ is called a natural \mathcal{L}_ξ -spline on τ if:

(i) $s \in C^{2p-2}(\mathbb{R})$;

(ii) $\mathcal{L}_\xi s(t) = 0, \quad \forall t \in (t_{j-1}, t_j), \quad \forall j \in \{0, 1, \dots, N+1\}$;

(iii) $\left(\frac{d}{dt} - |\xi| \right)^p s(t) = 0, \quad \forall t < t_0$, and $\left(\frac{d}{dt} + |\xi| \right)^p s(t) = 0, \quad \forall t > t_N$.

The space of all natural \mathcal{L}_ξ -splines on τ will be denoted by $\mathcal{S}_{p,\xi}(\tau)$.

Remark 1.3 For $\xi = 0$, this definition corresponds to the well-known natural polynomial splines of degree $2p - 1$. For $\xi \neq 0$, the use of adjoint operators on the two extreme intervals in condition (iii) is equivalent to the decay of the natural \mathcal{L}_ξ -spline s at $\pm\infty$. This differs from the standard natural conditions for Chebyshev splines (cf. Schumaker [15, p. 396]), which employ one and the same natural operator on both sides. It is of interest to note that in the case of semi-cardinal interpolation with \mathcal{L}_ξ -splines treated in [3], the splitting of the operator \mathcal{L}_ξ into its two adjoint factors on the left/right boundary pieces is a direct consequence of the Wiener-Hopf factorization technique.

The above arguments show that $S \in \mathcal{S}_p(\tau, n)$ implies $\widehat{S}_\xi \in \mathcal{S}_{p,\xi}(\tau), \forall \xi \in \mathbb{Z}^n$. Conversely, our results on Beppo Levi polyspline interpolation in section 3 will follow from the properties of the natural exponential \mathcal{L}_ξ -splines. The construction and necessary analysis of the natural \mathcal{L}_ξ -spline interpolation schemes are contained in section 2. An original contribution of this analysis is the use of radial basis representations to estimate the effect of the variable parameter ξ on the size of the Lagrange functions for natural \mathcal{L}_ξ -spline interpolation.

The extension of our results to the setting of data prescribed on concentric spheres will have to address significant technical differences. A further problem of interest would be to study the convergence properties of Beppo Levi polyspline interpolation.

2 Natural \mathcal{L}_ξ -spline interpolation

Note that the operator \mathcal{L}_ξ of (6) and the space $\mathcal{S}_{p,\xi}(\tau)$ of natural \mathcal{L}_ξ -splines can be defined not only for $\xi \in \mathbb{Z}^n$, but for any $\xi \in \mathbb{R}^n$. Therefore in this section we work with $\xi \in \mathbb{R}^n \setminus \{0\}$, the version of the results for $\xi = 0$ being well-known. We will also assume that $p \geq 2$ throughout the section.

Our first result asserts existence and uniqueness for the problem of natural \mathcal{L}_ξ -spline interpolation at the set of knots $\tau = \{t_0, t_1, \dots, t_N\}$.

Theorem 2.1 For any set of data values $\{y_0, \dots, y_N\} \subset \mathbb{R}$, there exists a unique natural \mathcal{L}_ξ -spline on τ , $s \in \mathcal{S}_{p,\xi}(\tau)$, such that

$$s(t_j) = y_j, \quad j \in \{0, 1, \dots, N\}. \quad (7)$$

We will derive the proof of this theorem from the following result.

Theorem 2.2 Let $\sigma \in \mathcal{S}_{p,\xi}(\tau)$ be any natural \mathcal{L}_ξ -spline on τ and let $\psi \in C^p(\mathbb{R})$ such that $\psi^{(m)} \in L^2(\mathbb{R})$, $\forall m \in \{0, \dots, p\}$, and $\psi(t_j) = 0$, $\forall j \in \{0, \dots, N\}$. Then the following ‘fundamental identity’ holds:

$$\int_{-\infty}^{\infty} \left(\frac{d}{dt} - |\xi| \right)^p \sigma(t) \left(\frac{d}{dt} - |\xi| \right)^p \overline{\psi(t)} dt = 0. \quad (8)$$

Another direct consequence of (8) is the variational characterization of the natural \mathcal{L}_ξ -spline s of Theorem 2.1. Although this will not be employed further, it may be of interest to mention it here.

Theorem 2.3 Given the set of values $\{y_0, \dots, y_N\} \subset \mathbb{R}$, let s be the unique natural \mathcal{L}_ξ -spline on τ satisfying the interpolation conditions (7). If $f \in C^p(\mathbb{R})$ is any other function such that $f^{(m)} \in L^2(\mathbb{R})$, $\forall m \in \{0, \dots, p\}$, and $f(t_j) = y_j$, $\forall j \in \{0, \dots, N\}$, then

$$\int_{-\infty}^{\infty} \left| \left(\frac{d}{dt} - |\xi| \right)^p s(t) \right|^2 dt < \int_{-\infty}^{\infty} \left| \left(\frac{d}{dt} - |\xi| \right)^p f(t) \right|^2 dt.$$

A related question at this point is whether the statements of the last two theorems remain true if the left natural operator $\left(\frac{d}{dt} - |\xi| \right)^p$ is replaced by its adjoint $\left(-\frac{d}{dt} - |\xi| \right)^p$. One way to see that the answer is positive is to go through the proof of Theorem 2.2 in subsection 2.1 and make the requisite changes. A faster way, however, is to use the next result that will also play a significant role in the arguments of section 3.

Lemma 2.4 Let $\sigma \in \mathcal{S}_{p,\xi}(\tau)$ be any natural \mathcal{L}_ξ -spline on τ and let $\psi \in C^p(\mathbb{R})$ such that $\psi^{(m)} \in L^2(\mathbb{R})$, $\forall m \in \{0, \dots, p\}$. Then

$$\begin{aligned} & \int_{-\infty}^{\infty} \left(\frac{d}{dt} - |\xi| \right)^p \sigma(t) \left(\frac{d}{dt} - |\xi| \right)^p \overline{\psi(t)} dt \\ &= \sum_{m=0}^p \binom{p}{m} |\xi|^{2(p-m)} \int_{-\infty}^{\infty} \sigma^{(m)}(t) \overline{\psi^{(m)}(t)} dt, \end{aligned} \quad (9)$$

where $\binom{p}{m} = \frac{p!}{m!(p-m)!}$ is the usual binomial coefficient.

The proof of this identity in subsection 2.2 works by showing that, after expanding the two brackets in the left-hand side, integrating term by term, and collecting terms of the same power of $|\xi|$, the odd power terms vanish. Since

replacing $\frac{d}{dt} - |\xi|$ by $\frac{d}{dt} + |\xi|$ in the left-hand side will only change the sign of the odd power terms, we deduce that, under the hypotheses of Lemma 2.4,

$$\begin{aligned} & \int_{-\infty}^{\infty} \left(\frac{d}{dt} + |\xi| \right)^p \sigma(t) \left(\frac{d}{dt} + |\xi| \right)^p \overline{\psi(t)} dt \\ &= \int_{-\infty}^{\infty} \left(\frac{d}{dt} - |\xi| \right)^p \sigma(t) \left(\frac{d}{dt} - |\xi| \right)^p \overline{\psi(t)} dt. \end{aligned}$$

Therefore Theorems 2.2 and 2.3 also hold with $\frac{d}{dt} + |\xi|$ in place of $\frac{d}{dt} - |\xi|$.

It can be noticed that, for a fixed ξ , Theorems 2.1, 2.2, and 2.3 follow along a rather classical route in spline theory, the only novelty being due to the adjoint boundary operators in Definition 1.2 of the natural \mathcal{L}_ξ -spline. The classical theory, however, does not cover the dependence of the natural \mathcal{L}_ξ -spline interpolation scheme on the variable parameter ξ , as will be needed in section 3. In order to study this dependence, for each $\xi \in \mathbb{Z}^n$ and $j \in \{0, 1, \dots, N\}$, let $L_{\xi,j} \in \mathcal{S}_{p,\xi}(\tau)$ be the unique *Lagrange function* determined by the interpolation conditions

$$L_{\xi,j}(t_j) = 1 \text{ and } L_{\xi,j}(t_k) = 0 \text{ for } k \in \{0, 1, \dots, N\} \setminus \{j\}. \quad (10)$$

Hence, if s is the natural \mathcal{L}_ξ -spline that interpolates the values y_0, y_1, \dots, y_N in Theorem 2.1, we have the Lagrange formula

$$s(t) = \sum_{j=0}^N y_j L_{\xi,j}(t), \quad \forall t \in \mathbb{R}. \quad (11)$$

The following theorem estimates the effect of the parameter ξ on the stability of the Lagrange scheme (11).

Theorem 2.5 *There exists a constant $C_0 = C_0(p, \tau) > 0$ such that, for all $j \in \{0, \dots, N\}$, $m \in \{0, \dots, 2p-2\}$, and all $\xi \in \mathbb{R}^n$ with $|\xi| \geq \frac{1}{2}$, we have:*

$$\left| \frac{d^m}{dt^m} L_{\xi,j}(t) \right| \leq C_0 (1 + |\xi|^m), \quad \forall t \in \mathbb{R}. \quad (12)$$

A similar statement for $m = 0$ and $t \in [t_0, t_N]$ was given by Kounchev [8, Lemma 3] in the context of an \mathcal{L}_ξ -spline interpolation scheme with different boundary conditions and an even integer p . The validity of the present result for all orders $m \in \{0, \dots, 2p-2\}$ is of crucial importance for the analysis of section 3. In subsection 2.3 we will employ an original method of proof of this result via ‘radial basis’ representations of the Lagrange functions $L_{\xi,j}$.

2.1 Proofs of Theorems 2.1, 2.2, and 2.3

We start by establishing the following auxiliary result.

Lemma 2.6 *If $\psi \in C^1(\mathbb{R})$ and $\psi' \in L^2(\mathbb{R})$, then there exists $C_\psi \geq 0$ such that*

$$|\psi(t)| \leq C_\psi \left(1 + |t|^{1/2}\right), \quad \forall t \in \mathbb{R}.$$

Proof. Combining the Leibniz-Newton formula $\psi(t) = \psi(0) + \int_0^t \psi'(u) du$ with the Cauchy-Schwarz inequality, we obtain for $t > 0$:

$$\begin{aligned} |\psi(t)| &\leq |\psi(0)| + \left(\int_0^t du\right)^{1/2} \left(\int_0^t |\psi'(u)|^2 du\right)^{1/2} \\ &\leq |\psi(0)| + t^{1/2} \left(\int_{\mathbb{R}} |\psi'(u)|^2 du\right)^{1/2}. \end{aligned}$$

The conclusion follows by making a similar estimate for $t < 0$ and letting $C_\psi := \max\left\{|\psi(0)|, \|\psi'\|_{L^2(\mathbb{R})}\right\}$. ■

Proof of Theorem 2.2. Let J_ξ denote the convergent integral of (8). Since $\left(\frac{d}{dt} - |\xi|\right)^p \sigma(t) = 0$ for $t \leq t_0$, the integration domain of J_ξ can be replaced by the interval $[t_0, \infty)$. Using integration by parts,

$$\begin{aligned} J_\xi &= \left[\left(\frac{d}{dt} - |\xi|\right)^p \sigma(t) \left(\frac{d}{dt} - |\xi|\right)^{p-1} \overline{\psi(t)} \right]_{t=t_0}^\infty \\ &\quad - \int_{t_0}^\infty \left(\frac{d}{dt} + |\xi|\right) \left(\frac{d}{dt} - |\xi|\right)^p \sigma(t) \left(\frac{d}{dt} - |\xi|\right)^{p-1} \overline{\psi(t)} dt. \end{aligned}$$

The expression in square brackets vanishes at t_0 due to the above condition on σ , and it also vanishes at ∞ due to the exponential decay of σ at ∞ and the fact that, by Lemma 2.6, the function $\left(\frac{d}{dt} - |\xi|\right)^{p-1} \overline{\psi(t)}$ has at most an algebraic growth at ∞ . Applying similar arguments successively,

$$J_\xi = (-1)^{p-1} \int_{t_0}^\infty \left(\frac{d}{dt} + |\xi|\right)^{p-1} \left(\frac{d}{dt} - |\xi|\right)^p \sigma(t) \left(\frac{d}{dt} - |\xi|\right) \overline{\psi(t)} dt.$$

Since $\sigma \in \mathcal{S}_{p,\xi}(\tau)$, for each $j \in \{0, \dots, N\}$ there exists a constant σ_j such that

$$\begin{aligned} &\left(\frac{d}{dt} + |\xi|\right)^{p-1} \left(\frac{d}{dt} - |\xi|\right)^p \sigma(t) \\ &= \sigma_j \left(\frac{d}{dt} + |\xi|\right)^{p-1} \left(\frac{d}{dt} - |\xi|\right)^p \left(t^{p-1} e^{-|\xi|t}\right) \\ &= \sigma_j (p-1)! (-2|\xi|)^p e^{-|\xi|t}, \quad \forall t \in (t_j, t_{j+1}), \end{aligned}$$

where $t_{N+1} := \infty$. Hence

$$J_\xi = -(p-1)! (2|\xi|)^p \sum_{j=0}^N \sigma_j \int_{t_j}^{t_{j+1}} e^{-|\xi|t} \left(\frac{d}{dt} - |\xi|\right) \overline{\psi(t)} dt.$$

Noting that $e^{-|\xi|t} \left(\frac{d}{dt} - |\xi| \right) \overline{\psi(t)} = \frac{d}{dt} \left(e^{-|\xi|t} \overline{\psi(t)} \right)$ and using the hypotheses $\psi(t_j) = 0, j \in \{0, 1, \dots, N\}$, as well as $\lim_{t \rightarrow \infty} e^{-|\xi|t} \psi(t) = 0$ (by Lemma 2.6), it follows that $J_\xi = 0$. ■

Proof of Theorem 2.1. Note that any function $s \in \mathcal{S}_{p,\xi}(\tau)$ is uniquely determined as the extension on \mathbb{R} of a function $s \in C^{2p-2}[t_0, t_N]$ which is in $\text{Ker}\mathcal{L}_\xi$ on any subinterval (t_{j-1}, t_j) , $j \in \{1, \dots, N\}$, and which satisfies the following endpoint conditions:

$$\begin{cases} \frac{d^m}{dt^m} \left(\frac{d}{dt} - |\xi| \right)^p s(t) \big|_{t=t_0} = 0, \\ \frac{d^m}{dt^m} \left(\frac{d}{dt} + |\xi| \right)^p s(t) \big|_{t=t_N} = 0, \end{cases} \quad m \in \{0, 1, \dots, p-2\}. \quad (13)$$

The restriction of such a function s to each subinterval (t_{j-1}, t_j) , $j \in \{1, \dots, N\}$, is determined by $2p$ coefficients. Therefore imposing on s the continuity conditions of class C^{2p-2} at each interior knot t_1, \dots, t_{N-1} , as well as the endpoint conditions (13) and the interpolation conditions (7), we obtain a system of $(2p-1)(N-1) + 2(p-1) + N + 1 = 2pN$ linear equations for as many coefficients.

To establish the nature of this system, we assume zero interpolation data: $y_j = 0, j \in \{0, 1, \dots, N\}$, in which case the system becomes homogeneous. Letting s be determined by an arbitrary solution of this homogeneous system and $\sigma = \psi := s$ in (8), we obtain $s(t) \in \text{Ker} \left(\frac{d}{dt} - |\xi| \right)^p$ for $t \in \mathbb{R}$. Since $s(t_j) = 0, j \in \{0, 1, \dots, N\}$, and $N+1 \geq p$, we deduce $s \equiv 0$. It follows that the above homogeneous system has only the trivial solution, which proves the conclusion of the theorem. ■

Proof of Theorem 2.3. Letting $\sigma := s$ and $\psi := f - s$ in Theorem 2.2,

$$\int_{-\infty}^{\infty} \left(\frac{d}{dt} - |\xi| \right)^p s(t) \left(\frac{d}{dt} - |\xi| \right)^p \overline{(f(t) - s(t))} dt = 0,$$

hence

$$\begin{aligned} 0 &\leq \int_{-\infty}^{\infty} \left| \left(\frac{d}{dt} - |\xi| \right)^p (f(t) - s(t)) \right|^2 dt \\ &= \int_{-\infty}^{\infty} \left| \left(\frac{d}{dt} - |\xi| \right)^p f(t) \right|^2 dt - \int_{-\infty}^{\infty} \left| \left(\frac{d}{dt} - |\xi| \right)^p s(t) \right|^2 dt. \end{aligned}$$

The inequality can become equality only if $f(t) - s(t) \in \text{Ker} \left(\frac{d}{dt} - |\xi| \right)^p$ for $t \in \mathbb{R}$. Since $f - s$ vanishes at the knots t_0, \dots, t_N , this implies $f \equiv s$. ■

2.2 Proof of Lemma 2.4

Let J_ξ denote the integral on the left-hand side of (9). Expanding the two brackets inside this integral and collecting all the terms that have the same power l of $|\xi|$, we obtain

$$J_\xi = \sum_{l=0}^{2p} (-1)^l |\xi|^l J_{\xi,l}, \quad (14)$$

where

$$J_{\xi,l} := \sum_{r=0}^l \binom{p}{l-r} \binom{p}{r} \int_{-\infty}^{\infty} \sigma^{(p-r)}(t) \overline{\psi^{(p-l+r)}(t)} dt.$$

By convention, the two binomial coefficients in the last formula are zero if $l-r > p$ or $r > p$, respectively. The lemma will follow by showing that in (14) the odd power terms vanish and each of the even power terms equals its correspondent term on the right-hand side of (9).

First, if $l = 2k + 1$, where $k \in \{0, 1, \dots, p-1\}$, then the term of index r of the sum $J_{\xi,l}$ contains the same product of binomial coefficients as the term of index $l-r$. Further, for $r \leq k$, integration by parts gives the relations

$$\begin{aligned} & \int_{-\infty}^{\infty} \sigma^{(p-r)}(t) \overline{\psi^{(p-l+r)}(t)} dt \\ &= - \int_{-\infty}^{\infty} \sigma^{(p-r-1)}(t) \overline{\psi^{(p-l+r+1)}(t)} dt \\ &= \dots = (-1)^{l-2r} \int_{-\infty}^{\infty} \sigma^{(p-l+r)}(t) \overline{\psi^{(p-r)}(t)} dt. \end{aligned}$$

Indeed, the boundary terms in each integration by parts are zero due to the exponential decay at $\pm\infty$ of $\sigma(t)$ and its derivatives, as well as to the algebraic growth at $\pm\infty$ of $\psi(t)$ and its derivatives of order at most $p-1$. The latter growth property is a consequence of Lemma 2.6 and the hypotheses on ψ . Since $l-2r$ is odd, we deduce that the terms of indices r and $l-r$ of the sum $J_{\xi,l}$ cancel, hence $J_{\xi,2k+1} = 0$.

Second, if $l = 2k$, where $k \in \{0, 1, \dots, p\}$, then, for $0 \leq r < k$, integrating by parts $k-r$ times yields

$$\begin{aligned} & \int_{-\infty}^{\infty} \sigma^{(p-r)}(t) \overline{\psi^{(p-2k+r)}(t)} dt \\ &= - \int_{-\infty}^{\infty} \sigma^{(p-r-1)}(t) \overline{\psi^{(p-2k+r+1)}(t)} dt \\ &= \dots = (-1)^{k-r} \int_{-\infty}^{\infty} \sigma^{(p-k)}(t) \overline{\psi^{(p-k)}(t)} dt, \end{aligned}$$

where the boundary terms in each integration by parts are zero by the same reasons as in the previous paragraph. Combining this with a similar argument for $k < r \leq 2k$, it follows that $J_{\xi,2k}$ is a multiple of the integral

$$\int_{-\infty}^{\infty} \sigma^{(p-k)}(t) \overline{\psi^{(p-k)}(t)} dt,$$

the actual factor that multiplies the integral being the value

$$(-1)^k \sum_{r=0}^{2k} (-1)^r \binom{p}{2k-r} \binom{p}{r} = \binom{p}{k}.$$

The last equality is a consequence of binomial expansions in the formal identity $(1-X)^p(1+X)^p = (1-X^2)^p$. Therefore (9) is true. ■

2.3 Proof of Theorem 2.5

For each $\xi \in \mathbb{R}^n \setminus \{0\}$, we first introduce the integrable fundamental solution φ_ξ of the operator (6), i.e. $\mathcal{L}_\xi \varphi_\xi = \delta_0$ (the Dirac mass at the origin).

Lemma 2.7 *The unique integrable fundamental solution of the operator \mathcal{L}_ξ is*

$$\varphi_\xi(t) = \frac{(-1)^p}{\gamma_p |\xi|^{2p-1}} e^{-|\xi||t|} \sum_{l=0}^{p-1} c_l |\xi|^l |t|^l, \quad t \in \mathbb{R},$$

where $c_l = \frac{(2p-2-l)!2^l}{l!(p-1-l)!}$, $\forall l \in \{0, \dots, p-1\}$, and $\gamma_p = (p-1)!2^{2p-1}$.

Proof. Note that an integrable function φ_ξ satisfies the distributional equation $\mathcal{L}_\xi \varphi_\xi = \delta_0$ if and only if its Fourier transform $\widehat{\varphi}_\xi(u) = \int_{\mathbb{R}} e^{-itu} \varphi_\xi(t) dt$ is

$$\widehat{\varphi}_\xi(u) = \frac{(-1)^p}{(u^2 + |\xi|^2)^p}, \quad u \in \mathbb{R}.$$

By Fourier inversion,

$$\varphi_\xi(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{itu} \widehat{\varphi}_\xi(u) du = \frac{(-1)^p}{\pi} \int_0^\infty \frac{\cos tu}{(u^2 + |\xi|^2)^p} du. \quad (15)$$

The last integral is evaluated in Watson [17, §6.16(1)] as

$$\int_0^\infty \frac{\cos tu}{(u^2 + |\xi|^2)^p} du = \frac{\pi^{1/2}}{(p-1)!} \left(\frac{|t|}{2|\xi|} \right)^{p-\frac{1}{2}} K_{p-\frac{1}{2}}(|\xi||t|), \quad t \neq 0,$$

where $K_{p-\frac{1}{2}}$ is the modified Bessel function expressible in finite terms [17, §3.71(12)] by

$$K_{p-\frac{1}{2}}(z) = \left(\frac{\pi}{2z} \right)^{1/2} e^{-z} \sum_{l=0}^{p-1} \frac{(p-1+l)!}{l!(p-1-l)!(2z)^l}.$$

Therefore

$$\int_0^\infty \frac{\cos tu}{(u^2 + |\xi|^2)^p} du = \frac{\pi e^{-|\xi||t|}}{(p-1)!(2|\xi|)^{2p-1}} \sum_{l=0}^{p-1} c_l |\xi|^l |t|^l,$$

with the coefficients c_l from the statement. The expression of φ_ξ follows. ■

For convenience, we will work with the following ‘normalization’ of φ_ξ :

$$\widetilde{\varphi}_\xi(t) := (-1)^p \gamma_p |\xi|^{2p-1} \varphi_\xi(t) = e^{-|\xi||t|} \sum_{l=0}^{p-1} c_l |\xi|^l |t|^l, \quad t \in \mathbb{R}. \quad (16)$$

For each $\xi \in \mathbb{R}^n \setminus \{0\}$ and $j \in \{0, 1, \dots, N\}$, define the function $L_{\xi,j}$ (independently of Theorem 2.1) by the ‘radial basis function’ (RBF) representation

$$L_{\xi,j}(t) = \sum_{k=0}^N a_{jk} \tilde{\varphi}_{\xi}(t - t_k), \quad t \in \mathbb{R}, \quad (17)$$

where the set of coefficients $\{a_{jk} : k = 0, \dots, N\}$ is determined uniquely in the next lemma from the system of interpolation conditions (10). Let

$$M_{\xi} := (\tilde{\varphi}_{\xi}(t_j - t_k))_{j,k=0}^N$$

be the matrix of this interpolation system under representation (17).

Lemma 2.8 *For every $\xi \in \mathbb{R}^n \setminus \{0\}$, the matrix M_{ξ} is positive definite, hence nonsingular. Accordingly, for each $j \in \{0, \dots, N\}$, the function $L_{\xi,j}$ defined by (17) and (10) is the Lagrange function in $\mathcal{S}_{p,\xi}(\tau)$ arising from Theorem 2.1.*

Proof. Writing the Fourier inversion formula (15) in the form

$$\tilde{\varphi}_{\xi}(t) = \frac{\gamma_p |\xi|^{2p-1}}{2\pi} \int_{\mathbb{R}} \frac{e^{itu}}{(u^2 + |\xi|^2)^p} du, \quad t \in \mathbb{R},$$

we obtain, for any column vector $\mathbf{v} = (v_0, \dots, v_N)^T \in \mathbb{R}^{N+1}$,

$$\begin{aligned} \mathbf{v}^T M_{\xi} \mathbf{v} &= \sum_{j=0}^N \sum_{k=0}^N \tilde{\varphi}_{\xi}(t_j - t_k) v_j v_k \\ &= \frac{\gamma_p |\xi|^{2p-1}}{2\pi} \int_{\mathbb{R}} \frac{\left| \sum_{k=0}^N v_k e^{it_k u} \right|^2}{(u^2 + |\xi|^2)^p} du \geq 0. \end{aligned}$$

Further, since the functions $e^{it_k u}$, for $k \in \{0, \dots, N\}$, are linearly independent (their Wronskian being the multiple of a Vandermonde determinant), it follows that the above inequality is strict if $\mathbf{v} \neq 0$. Therefore the matrix M_{ξ} is positive definite and nonsingular.

For the last part of the lemma, it is sufficient to prove that $L_{\xi,j}$ defined by (17) and (10) is in $\mathcal{S}_{p,\xi}(\tau)$. Differentiating the above Fourier inversion formula shows that $\tilde{\varphi}_{\xi} \in C^{2p-2}(\mathbb{R})$, so the linear combination (17) is in $C^{2p-2}(\mathbb{R})$. On the other hand, for any non-negative integer l , any knot $t_k \in \tau$, and any parameter ξ , the following formula shows that the translate of an exponential polynomial is spanned by exponential polynomials of the same type, namely

$$(t - t_k)^l e^{\pm |\xi|(t - t_k)} = \sum_{r=0}^l a_r(l, t_k, |\xi|) t^r e^{\pm |\xi|t},$$

for some coefficients $a_r = a_r(l, t_k, |\xi|)$ (cf. [16]). We deduce that the form (17) of $L_{\xi,j}$ also satisfies conditions (ii) and (iii) of Definition 1.2, as required. The lemma is proved. ■

Lemma 2.9 *There exists a constant $\mu = \mu(p, \tau) > 0$ such that, for any $\xi \in \mathbb{R}^n$ with $|\xi| \geq \mu$ and any eigenvalue λ of M_ξ , we have $\lambda \geq \frac{1}{2}c_0$, where $c_0 = \frac{(2p-2)!}{(p-1)!}$.*

Proof. We follow an idea from Narcowich-Ward [11] and Schaback [14]. Note that the diagonal entries of M_ξ are all equal to $\tilde{\varphi}_\xi(0) = c_0$. We show that M_ξ is diagonally dominant for sufficiently large $|\xi|$. Let

$$\psi(u) := e^{-u} \sum_{l=0}^{p-1} c_l u^l, \quad u \geq 0,$$

so that $\tilde{\varphi}_\xi(t) = \psi(|\xi||t|)$, $\forall t \in \mathbb{R}$. Since $\lim_{u \rightarrow \infty} \psi(u) = 0$, there exists $\delta > 0$ depending only on p and N such that

$$\psi(u) \leq \frac{c_0}{2N}, \quad \forall u \geq \delta.$$

Let $\mu := \delta / \min_{j \neq k} |t_j - t_k|$. Then for all ξ with $|\xi| \geq \mu$ and all $j \neq k$, we have $|\xi||t_j - t_k| \geq \delta$, from which

$$\rho := \max_{0 \leq j \leq N} \sum_{\substack{k=0 \\ k \neq j}}^N \tilde{\varphi}_\xi(t_j - t_k) \leq \frac{c_0}{2}.$$

It follows that the matrix M_ξ is diagonally dominant and Gershgorin's circle theorem [7, p. 395] implies $\lambda \geq \tilde{\varphi}_\xi(0) - \rho \geq \frac{1}{2}c_0$, for any eigenvalue λ of M_ξ . ■

Proof of Theorem 2.5. First we show that there exists a constant $A = A(p, \tau)$ such that the coefficients a_{jk} of the RBF representation (17) satisfy

$$|a_{jk}| \leq A, \quad \forall j, k \in \{0, \dots, N\}, \quad \forall |\xi| \geq 1/2. \quad (18)$$

Note that, if $\mathbf{a}_j := (a_{j0}, \dots, a_{jN})^T \in \mathbb{R}^{N+1}$ and \mathbf{e}_j is the j th column of the identity matrix of order $N+1$, then $\mathbf{a}_j = M_\xi^{-1} \mathbf{e}_j$, i.e. \mathbf{a}_j is the j th column of the inverse matrix M_ξ^{-1} . Hence, for $|\xi| \geq \mu$ and any j, k , Lemma 2.9 implies

$$|a_{jk}| \leq \|\mathbf{a}_j\|_2 = \|M_\xi^{-1} \mathbf{e}_j\|_2 \leq \|M_\xi^{-1}\|_2 = \frac{1}{\min |\lambda|} \leq \frac{2}{c_0},$$

where $\|\mathbf{a}_j\|_2$ is the Euclidean norm in \mathbb{R}^{N+1} , $\|M_\xi^{-1}\|_2$ is the induced matrix norm, and the minimum is taken over all eigenvalues λ of M_ξ . If $\frac{1}{2} \geq \mu$, then (18) holds with $A := \frac{2}{c_0}$. If $\frac{1}{2} < \mu$, then for each $\xi \in \mathbb{R}^n$ with $\frac{1}{2} \leq |\xi| \leq \mu$, we use Cramer's rule to express each coefficient a_{jk} as the ratio of two determinants, the denominator being $\det M_\xi$. The numerator is the determinant of the matrix obtained from M_ξ by replacing its j th column with \mathbf{e}_k . Since the entries $\tilde{\varphi}_\xi(t_j - t_k)$ of M_ξ depend continuously on ξ , it follows that there exists a constant $b_0 = b_0(p, \tau) > 0$, such that $|a_{jk}| \leq b_0$ for all j, k , and all ξ with $\frac{1}{2} \leq |\xi| \leq \mu$. Therefore (18) is obtained by letting $A := \max \left\{ b_0, \frac{2}{c_0} \right\}$.

Since the values $\tilde{\varphi}_\xi(t)$ of (16) are positive and bounded above by a constant for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, from (17) and (18) we deduce that (12) holds for $m = 0$. The corresponding estimates for the derivatives of $L_{\xi,j}$ follow in the same way, since (16) implies

$$\tilde{\varphi}_\xi^{(m)}(t) = |\xi|^m e^{-|\xi||t|} \sum_{l=0}^{p-1} c_{m,l} |\xi|^l |t|^l, \quad t \in \mathbb{R}, \quad m \in \{1, \dots, 2p-2\}, \quad (19)$$

for certain coefficients $c_{m,l}$. This completes the proof. ■

3 Transfinite interpolation with Beppo Levi polysplines

For any non-negative integer r , let $W^r(\mathbb{T}^n)$ be the space of all continuous functions $f : \mathbb{R}^n \rightarrow \mathbb{C}$ which are 2π -periodic in each variable and satisfy the condition

$$\|f\|_r := \sum_{\xi \in \mathbb{Z}^n} \left| \hat{f}_\xi \right| (1 + |\xi|)^r < \infty,$$

where $\hat{f}_\xi := \int_{\mathbb{T}^n} e^{-i\langle \xi, y \rangle} f(y) dy$, for $\xi \in \mathbb{Z}^n$, are the Fourier coefficients of f . In particular, $W^0(\mathbb{T}^n)$ is the Wiener algebra of functions with absolutely convergent Fourier series. Note that $0 \leq r_1 \leq r_2$ implies $\|f\|_{r_1} \leq \|f\|_{r_2}$, hence $W^{r_2}(\mathbb{T}^n) \subset W^{r_1}(\mathbb{T}^n)$. It is also straightforward that $C^k(\mathbb{T}^n) \subset W^r(\mathbb{T}^n) \subset C^r(\mathbb{T}^n)$, where k is the least integer greater than or equal to $r + \frac{n+1}{2}$.

Now recall the setting of the Introduction, in which $p \geq 2$. First, we state the existence and uniqueness of transfinite interpolation with Beppo Levi polysplines of order p on strips.

Theorem 3.1 *For any set of data functions $f_j \in W^{2p-2}(\mathbb{T}^n)$, $j \in \{0, \dots, N\}$, there exists a unique Beppo Levi polyspline $S \in \mathcal{S}_p(\tau, n)$ such that*

$$S(t_j, y) = f_j(y), \quad j \in \{0, 1, \dots, N\}, \quad y \in \mathbb{T}^n. \quad (20)$$

Let $\mathcal{B}_p := \mathcal{B}_p(\mathbb{R} \times \mathbb{T}^n)$ be the space of all functions $F \in C^p(\mathbb{R}^{n+1})$ that are 2π -periodic in each of the last n variables and satisfy, in usual multi-index notation,

$$\partial^\alpha F \in L^2(\mathbb{R} \times \mathbb{T}^n), \quad \forall |\alpha| = p.$$

On \mathcal{B}_p we define the following semi-inner product and induced seminorm:

$$\begin{aligned} \langle F, G \rangle_{\mathcal{B}_p} &:= \int_{\mathbb{R} \times \mathbb{T}^n} \sum_{|\alpha|=p} \frac{p!}{\alpha!} \partial^\alpha F \overline{\partial^\alpha G} dt dy, \quad \forall F, G \in \mathcal{B}_p, \\ \|F\|_{\mathcal{B}_p} &:= \langle F, F \rangle_{\mathcal{B}_p}^{1/2}, \quad \forall F \in \mathcal{B}_p. \end{aligned} \quad (21)$$

Note that $\mathcal{S}_p(\tau, n) \subset \mathcal{B}_p$ for $p \geq 2$, and that $\|F\|_{\mathcal{B}_p}^2$ coincides with (5).

The following orthogonality result is the polyspline analog of Theorem 2.2.

Theorem 3.2 *Let $S \in \mathcal{S}_p(\tau, n)$ be any Beppo Levi polyspline as in Definition 1.1. If $G \in \mathcal{B}_p$ satisfies $G(t_j, y) = 0$, $\forall j \in \{0, 1, \dots, N\}$, $\forall y \in \mathbb{T}^n$, then*

$$\langle S, G \rangle_{\mathcal{B}_p} = 0.$$

This ‘fundamental identity’ is eventually used to establish the characterization of the Beppo Levi polyspline S of Theorem 3.1 as minimizer of the Duchon type seminorm $\|\cdot\|_{\mathcal{B}_p}$ subject to the transfinite interpolation conditions (20).

Theorem 3.3 *Given the set of data functions $\{f_0, f_1, \dots, f_N\} \subset W^{2p-2}(\mathbb{T}^n)$, let $S \in \mathcal{S}_p(\tau \times \mathbb{T}^n)$ be the Beppo Levi polyspline that satisfies conditions (20). If $F \in \mathcal{B}_p$ is any other function satisfying $F(t_j, y) = f_j(y)$, $\forall j \in \{0, \dots, N\}$, $\forall y \in \mathbb{T}^n$, then*

$$\|S\|_{\mathcal{B}_p} < \|F\|_{\mathcal{B}_p}.$$

The proofs below will employ standard notation. In particular, $\mathbb{Z}_+ := \{0, 1, \dots\}$ and, for a multi-index $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$, we let $|\beta| := \beta_1 + \dots + \beta_n$, $\beta! := \beta_1! \dots \beta_n!$, and $\partial_y^\beta := \frac{\partial^{\beta_1}}{\partial y_1^{\beta_1}} \dots \frac{\partial^{\beta_n}}{\partial y_n^{\beta_n}}$.

3.1 Proof of Theorem 3.1

To construct the natural polyspline S with the properties stated in Theorem 3.1, we start from the absolutely convergent Fourier series representations

$$f_j(y) = \sum_{\xi \in \mathbb{Z}^n} \widehat{f}_{j,\xi} e^{i\langle \xi, y \rangle}, \quad j \in \{0, 1, \dots, N\}, \quad y \in \mathbb{T}^n,$$

where $\widehat{f}_{j,\xi}$, for $\xi \in \mathbb{Z}^n$, are the Fourier coefficients of $f_j \in W^{2p-2}(\mathbb{T}^n)$. For each $\xi \in \mathbb{Z}^n$, by Theorem 2.1 and its classical version for $\xi = 0$, let $\widehat{S}_\xi \in \mathcal{S}_{p,\xi}(\tau)$ be the unique natural \mathcal{L}_ξ -spline on τ , such that

$$\widehat{S}_\xi(t_j) = \widehat{f}_{j,\xi}, \quad j \in \{0, 1, \dots, N\}.$$

By the Lagrange formula (11), we have

$$\widehat{S}_\xi(t) = \sum_{j=0}^N \widehat{f}_{j,\xi} L_{\xi,j}(t), \quad t \in \mathbb{R}. \quad (22)$$

We will prove that the function S defined by

$$S(t, y) := \sum_{\xi \in \mathbb{Z}^n} \widehat{S}_\xi(t) e^{i\langle \xi, y \rangle}, \quad y \in \mathbb{T}^n, \quad t \in \mathbb{R}, \quad (23)$$

possesses all the properties required by Theorem 3.1.

First, we show that $S \in C^{2p-2}(\mathbb{R} \times \mathbb{T}^n)$ and S satisfies (20). Let $m \in \mathbb{Z}_+$, $\beta \in \mathbb{Z}_+^n$, and $\alpha = (m, \beta)$, such that $|\alpha| = m + |\beta| \leq 2p - 2$. For $t \in [t_0, t_N]$, we substitute (22) into (23) and use the estimate (12) to obtain formally

$$\begin{aligned} |\partial^\alpha S(t, y)| &\leq \sum_{\xi \in \mathbb{Z}^n} \sum_{j=0}^N \left| \widehat{f}_{j,\xi} \right| \left| \partial_t^m \partial_y^\beta L_{\xi,j}(t) e^{i\langle \xi, y \rangle} \right| \\ &\leq C_0 \sum_{\xi \in \mathbb{Z}^n} \sum_{j=0}^N \left| \widehat{f}_{j,\xi} \right| (1 + |\xi|^m) |\xi|^{|\beta|} \\ &\leq 2C_0 \sum_{j=0}^N \|f_j\|_{|\alpha|}, \quad \forall y \in \mathbb{T}^n. \end{aligned}$$

For $t \notin [t_0, t_N]$, we use the same estimates for all terms corresponding to $\xi \in \mathbb{Z}^n \setminus \{0\}$ in the first line of the above display, while the modulus of the term corresponding to $\xi = 0$ is bounded above by

$$\sum_{j=0}^N \left| \widehat{f}_{j,0} \right| \left| \partial_t^m \partial_y^\beta L_{0,j}(t) \right|.$$

This expression vanishes if $|\beta| \geq 1$, or if $\beta = 0$ and $m \geq p$, since $L_{0,j}$ is a polynomial of degree at most $p-1$ outside the interval $[t_0, t_N]$. In the remaining case $\beta = 0$ and $0 \leq m \leq p-1$, we use the polynomial growth estimate

$$\left| \frac{d^m}{dt^m} L_{0,j}(t) \right| \leq C_1 \left(1 + |t|^{p-1-m} \right), \quad \forall t \notin [t_0, t_N], \forall j \in \{0, \dots, N\},$$

with the constant C_1 depending only on p and τ . Altogether, these estimates show that the series (23) is absolutely and uniformly convergent on compact sets in $\mathbb{R} \times \mathbb{T}^n$ and can be differentiated termwise up to the total order $2p-2$. We also obtain (20) from the Lagrange conditions (10) satisfied by $L_{\xi,j}$.

Next, to show that S is polyharmonic of order p on each of the strips Ω_j , $j \in \{0, \dots, N+1\}$, we use the following result.

Lemma 3.4 *Suppose that a series of functions is absolutely and uniformly convergent on compact sets in the open domain Ω and each term of the series is polyharmonic of order p on Ω . Then the sum of the series is also polyharmonic of order p on Ω .*

This is well-known in the harmonic case $p = 1$. For $p \geq 2$, it was proved by Nicolesco [12, p. 23] based on integral mean representations of polyharmonic functions. In our case, each term of (23) satisfies

$$\begin{aligned} \Delta^p \left[\widehat{S}_\xi(t) e^{i\langle \xi, y \rangle} \right] &= \left(\frac{\partial^2}{\partial t^2} + \sum_{\nu=1}^d \frac{\partial^2}{\partial y_\nu^2} \right)^p \left[\widehat{S}_\xi(t) e^{i\langle \xi, y \rangle} \right] \\ &= e^{i\langle \xi, y \rangle} \left(\frac{d^2}{dt^2} - |\xi|^2 \right)^p \widehat{S}_\xi(t) = 0, \end{aligned}$$

for $(t, y) \in \Omega_j$, $j \in \{0, \dots, N+1\}$, since $\widehat{S}_\xi \in \mathcal{S}_{p,\xi}(\tau)$. Therefore S is piecewise polyharmonic as required.

It remains to prove that S satisfies the Beppo Levi condition (4). Let $m \in \mathbb{Z}_+$, $\beta \in \mathbb{Z}_+^n$, and $\alpha = (m, \beta)$, such that $|\alpha| = m + |\beta| = p$. It is sufficient to prove that the termwise partial derivative ∂^α of series (23) is a Cauchy series in $L^2((\mathbb{R} \setminus [t_0, t_N]) \times \mathbb{T}^n)$, which implies convergence to its sum $\partial^\alpha S(t, y)$ in $L^2((\mathbb{R} \setminus [t_0, t_N]) \times \mathbb{T}^n)$. Note that the partial derivative ∂^α of order p of the term corresponding to $\xi = 0$ in (23) vanishes due to the natural conditions satisfied by \widehat{S}_ξ outside the interval $[t_0, t_N]$. If $\xi \in \mathbb{Z}^n \setminus \{0\}$ and $0 \leq j \leq N$, relations (16) and (17) give the following representation of $L_{\xi,j}$ for $t \leq t_0$:

$$L_{\xi,j}(t) = \sum_{k=0}^N a_{jk} e^{|\xi|(t-t_k)} \sum_{l=0}^{p-1} c_l |\xi|^l (t_k - t)^l.$$

Hence, for $m = 0$ and $|\beta| = p$, Minkowski's inequality and (18) imply

$$\begin{aligned} & \left(\int_{-\infty}^{t_0} \int_{\mathbb{T}^n} \left| \partial_y^\beta L_{\xi,j}(t) e^{i\langle \xi, y \rangle} \right|^2 dy dt \right)^{1/2} \\ & \leq |\xi|^p \sum_{k=0}^N |a_{jk}| \sum_{l=0}^{p-1} c_l |\xi|^l \left(\int_{-\infty}^{t_0} (t_k - t)^{2l} e^{2|\xi|(t-t_k)} dt \right)^{1/2} \\ & \leq A |\xi|^p \sum_{k=0}^N \sum_{l=0}^{p-1} c_l |\xi|^l \left(\int_{-\infty}^{t_k} (t_k - t)^{2l} e^{2|\xi|(t-t_k)} dt \right)^{1/2}. \end{aligned}$$

Since, by changing the variable, the last integral becomes Euler's integral

$$\int_0^\infty u^{2l} e^{-2|\xi|u} du = \frac{(2l)!}{(2|\xi|)^{2l+1}},$$

we obtain

$$\left(\int_{-\infty}^{t_0} \int_{\mathbb{T}^n} \left| \partial_y^\beta L_{\xi,j}(t) e^{i\langle \xi, y \rangle} \right|^2 dy dt \right)^{1/2} \leq C_2 |\xi|^{p-\frac{1}{2}},$$

for a constant C_2 depending only on p and τ . For $m \geq 1$ and $m + |\beta| = p$, a similar argument based on the RBF derivative (19) gives

$$\left(\int_{-\infty}^{t_0} \int_{\mathbb{T}^n} \left| \partial_t^m \partial_y^\beta L_{\xi,j}(t) e^{i\langle \xi, y \rangle} \right|^2 dy dt \right)^{1/2} \leq C_2 |\xi|^{p-\frac{1}{2}},$$

after increasing C_2 if necessary. Therefore, if $\alpha \in \mathbb{Z}_+^{n+1}$ is any multi-index with $|\alpha| = p$ and \mathcal{F} is any finite subset of $\mathbb{Z}^n \setminus \{0\}$, another application of Minkowski's

inequality gives

$$\begin{aligned}
& \left(\int_{-\infty}^{t_0} \int_{\mathbb{T}^n} \left| \sum_{\xi \in \mathcal{F}} \sum_{j=0}^n \widehat{f}_{j,\xi} \partial^\alpha L_{\xi,j}(t) e^{i\langle \xi, y \rangle} \right|^2 dy dt \right)^{1/2} \\
& \leq \sum_{\xi \in \mathcal{F}} \sum_{j=0}^n \left| \widehat{f}_{j,\xi} \right| \left(\int_{-\infty}^{t_0} \int_{\mathbb{T}^n} \left| \partial^\alpha L_{\xi,j}(t) e^{i\langle \xi, y \rangle} \right|^2 dy dt \right)^{1/2} \\
& \leq C_2 \sum_{\xi \in \mathcal{F}} \sum_{j=0}^n \left| \widehat{f}_{j,\xi} \right| |\xi|^{p-\frac{1}{2}}.
\end{aligned}$$

Since $f_j \in W^{2p-2}(\mathbb{T}^n)$ and $p - \frac{1}{2} < 2p - 2$ for $2 \leq p$, we deduce that any termwise partial derivative of total order p of the series (23) is a Cauchy series in $L^2((-\infty, t_0) \times \mathbb{T}^n)$. By similar arguments, this conclusion also holds for the interval (t_N, ∞) in place of $(-\infty, t_0)$, which completes the proof that S satisfies the Beppo-Levi condition (4).

The uniqueness of a natural polyspline S satisfying the conditions of Theorem 3.1 follows directly from the uniqueness of natural \mathcal{L}_ξ -splines established in Theorem 2.1, as in the proof of uniqueness of [3, Theorem 15]. ■

3.2 Proofs of Theorems 3.2 and 3.3

The proof of Theorem 3.2 will employ Parseval type representations of the semi-inner product (21) with respect to the y variable. For any $F \in \mathcal{B}_p$ and $t \in \mathbb{R}$, define the Fourier coefficients of F with respect to its last n variables by

$$\widehat{F}_\xi(t) := \int_{\mathbb{T}^n} e^{-i\langle \xi, y \rangle} F(t, y) dy, \quad \xi \in \mathbb{Z}^n. \quad (24)$$

Lemma 3.5 *Let $F, G \in \mathcal{B}_p$. If $m \in \mathbb{Z}_+$, $\beta \in \mathbb{Z}_+^n$, and $\alpha = (m, \beta)$ satisfy $|\alpha| = m + |\beta| = p$, then the following identities hold with absolutely convergent integrals and series:*

$$\int_{\mathbb{T}^n} |\partial^\alpha F(t, y)|^2 dy = \sum_{\xi \in \mathbb{Z}^n} \xi^{2\beta} \left| \frac{d^m}{dt^m} \widehat{F}_\xi(t) \right|^2, \quad (25)$$

$$\int_{\mathbb{T}^n} \partial^\alpha F(t, y) \overline{\partial^\alpha G(t, y)} dy = \sum_{\xi \in \mathbb{Z}^n} \xi^{2\beta} \frac{d^m}{dt^m} \widehat{F}_\xi(t) \overline{\frac{d^m}{dt^m} \widehat{G}_\xi(t)}, \quad (26)$$

$$\int_{\mathbb{R}} \int_{\mathbb{T}^n} |\partial^\alpha F(t, y)|^2 dy dt = \sum_{\xi \in \mathbb{Z}^n} \xi^{2\beta} \int_{\mathbb{R}} \left| \frac{d^m}{dt^m} \widehat{F}_\xi(t) \right|^2 dt, \quad (27)$$

$$\int_{\mathbb{R}} \int_{\mathbb{T}^n} \partial^\alpha F(t, y) \overline{\partial^\alpha G(t, y)} dy dt = \sum_{\xi \in \mathbb{Z}^n} \xi^{2\beta} \int_{\mathbb{R}} \frac{d^m}{dt^m} \widehat{F}_\xi(t) \overline{\frac{d^m}{dt^m} \widehat{G}_\xi(t)} dt. \quad (28)$$

Proof. Differentiating with respect to t in (24), then integrating by parts with respect to each component of y and using the periodicity of F in y and the hypothesis $F \in C^p(\mathbb{R}^{n+1})$, we obtain

$$(i\xi)^\beta \frac{d^m}{dt^m} \widehat{F}_\xi(t) = \int_{\mathbb{T}^n} e^{-i\langle \xi, y \rangle} \partial_y^\beta \partial_t^m F(t, y) dy, \quad \xi \in \mathbb{Z}^n,$$

where $\xi \neq 0$ if $\beta \neq 0$. Since $\partial_y^\beta \partial_t^m F(t, \cdot) \in C(\mathbb{T}^n) \subset L^2(\mathbb{T}^n)$, the sequence of Fourier coefficients of $\partial_y^\beta \partial_t^m F(t, \cdot)$ on the left-hand side of the above display is square summable and the Parseval relations (25) and (26) hold. Further, $\partial_y^\beta \partial_t^m F = \partial^\alpha F \in L^2(\mathbb{R} \times \mathbb{T}^n)$ shows that formula (27) is obtained by integrating (25) and invoking Fubini's theorem in the right-hand side. Similarly, (28) follows by integrating (26), the use of Fubini's theorem being allowed due to the estimate

$$\begin{aligned} & \sum_{\xi \in \mathbb{Z}^n} \xi^{2\beta} \int_{\mathbb{R}} \left| \frac{d^m}{dt^m} \widehat{F}_\xi(t) \frac{d^m}{dt^m} \overline{\widehat{G}_\xi(t)} \right| dt \\ & \leq \left(\sum_{\xi \in \mathbb{Z}^n} \xi^{2\beta} \int_{\mathbb{R}} \left| \frac{d^m}{dt^m} \widehat{F}_\xi(t) \right|^2 dt \right)^{1/2} \left(\sum_{\xi \in \mathbb{Z}^n} \xi^{2\beta} \int_{\mathbb{R}} \left| \frac{d^m}{dt^m} \widehat{G}_\xi(t) \right|^2 dt \right)^{1/2} \\ & = \left(\int_{\mathbb{R}} \int_{\mathbb{T}^n} |\partial^\alpha F(t, y)|^2 dy dt \right)^{1/2} \left(\int_{\mathbb{R}} \int_{\mathbb{T}^n} |\partial^\alpha G(t, y)|^2 dy dt \right)^{1/2} < \infty, \end{aligned}$$

which is a consequence of two Schwarz inequalities and formula (27). ■

Proof of Theorem 3.2. Due to the absolute convergence of the integrals and the series of formula (28) with $F := S$, we have

$$\begin{aligned} \langle S, G \rangle_{\mathcal{B}_p} &= \int_{\mathbb{R}} \int_{\mathbb{T}^n} \sum_{|\alpha|=p} \frac{p!}{\alpha!} \partial^\alpha S(t, y) \overline{\partial^\alpha G(t, y)} dy dt \\ &= \sum_{\xi \in \mathbb{Z}^n} \sum_{|\alpha|=p} \frac{p!}{\alpha!} \xi^{2\beta} \int_{\mathbb{R}} \frac{d^m}{dt^m} \widehat{S}_\xi(t) \frac{d^m}{dt^m} \overline{\widehat{G}_\xi(t)} dt, \end{aligned}$$

where $\alpha = (m, \beta)$, as in Lemma 3.5. Next, replacing $\sum_{|\alpha|=p}$ by the double sum $\sum_{m=0}^p \sum_{|\beta|=p-m}$ and using $\alpha! = m!\beta!$ and the multinomial identity

$$\sum_{|\beta|=p-m} \frac{(p-m)!}{\beta!} \xi^{2\beta} = |\xi|^{2(p-m)},$$

we obtain

$$\langle S, G \rangle_{\mathcal{B}_p} = \sum_{\xi \in \mathbb{Z}^n} J_\xi,$$

where

$$J_\xi := \sum_{m=0}^p \binom{p}{m} |\xi|^{2(p-m)} \int_{-\infty}^{\infty} \frac{d^m}{dt^m} \widehat{S}_\xi(t) \frac{d^m}{dt^m} \overline{\widehat{G}_\xi(t)} dt.$$

We now claim that the hypotheses of Lemma 2.4 and Theorem 2.2 are verified for $\sigma := \widehat{S}_\xi$ and $\psi := \widehat{G}_\xi$. Indeed, we have seen in the Introduction that $S \in \mathcal{S}_p(\tau, n)$ implies $\widehat{S}_\xi \in \mathcal{S}_{p,\xi}(\tau)$, $\forall \xi \in \mathbb{Z}^n$. On the other hand, $G \in \mathcal{B}_p$ implies $\widehat{G}_\xi \in C^p(\mathbb{R})$ and, by (27), $\widehat{G}_\xi^{(m)} \in L^2(\mathbb{R})$, $\forall m \in \{0, \dots, p\}$, $\forall \xi \in \mathbb{Z}^n \setminus \{0\}$. The remaining condition $\widehat{G}_\xi(t_j) = 0$, $\forall j \in \{0, \dots, N\}$ follows from $G(t_j, y) = 0$, $\forall j \in \{0, \dots, N\}$, $\forall y \in \mathbb{T}^n$. Therefore Lemma 2.4 and Theorem 2.2 show that

$$J_\xi = \int_{-\infty}^{\infty} \left(\frac{d}{dt} - |\xi| \right)^p \widehat{S}_\xi(t) \left(\frac{d}{dt} - |\xi| \right)^p \overline{\widehat{G}_\xi(t)} dt = 0, \quad \forall \xi \in \mathbb{Z}^n,$$

which completes the proof. ■

Proof of Theorem 3.3. This follows in a standard way now by setting $G := F - S$ in Theorem 3.2, to obtain $\langle S, F - S \rangle_{\mathcal{B}_p} = 0$, hence $0 \leq \|F - S\|_{\mathcal{B}_p}^2 = \|F\|_{\mathcal{B}_p}^2 - \|S\|_{\mathcal{B}_p}^2$. Equality is only possible if $\|F - S\|_{\mathcal{B}_p} = 0$, i.e. if $F - S$ is in the null space of the seminorm (21). Since $p \geq 2$, this amounts to $\partial^\alpha (F - S) \equiv 0$ for all multi-indices $\alpha \in \mathbb{Z}_+^{n+1}$ with $|\alpha| = p$. Taking $\alpha = (0, \beta)$ with $\beta \in \mathbb{Z}_+^n$, it follows that the Fourier coefficients of $F - S$ with respect to the last n variables satisfy $\widehat{(F - S)}_\xi(t) = 0$ for all $\xi \in \mathbb{Z}^n \setminus \{0\}$, $t \in \mathbb{R}$. Further, taking $\alpha = (p, 0)$, we find that the remaining Fourier coefficient for $\xi = 0$ satisfies $\frac{d^p}{dt^p} \widehat{(F - S)}_0(t) = 0$, $\forall t \in \mathbb{R}$. Hence $(F - S)(t, y)$ is simply a polynomial of degree at most $p - 1$ in t . Since $N \geq p - 1$ and $(F - S)(t_j, y) = 0$ for all $j \in \{0, 1, \dots, N\}$ and $y \in \mathbb{T}^n$, we obtain $F - S \equiv 0$, Q.E.D. ■

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